

Convergence of basis expansions

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1 Introduction

The basis expansion technique[1, 2, 3, 4, 5] has been used for many years to find approximate eigenmodes in bunched beam instability problems. In general terms, one reduces the linearized Vlasov equation to an eigenvalue problem in one spatial (τ) and one momentum (v) variable,

$$\lambda f(\tau, v) = L_{op}[f(\tau, v)]. \quad (1)$$

In equation (1) λ is the eigenvalue, $f(\tau, v)$ is its eigenfunction, and L_{op} is a linear integro-differential operator. For all but the simplest cases [6, 7, 8, 9] no exact solutions are known. The basis expansion technique involves choosing a complete set of basis functions $g_n(\tau, v)$ and a weighting function $W(\tau, v)$ which satisfy an orthogonality relation

$$\int g_n^*(\tau, v) g_m(\tau, v) W(\tau, v) d\tau dv = \delta_m^n. \quad (2)$$

In equation (2) the integral is over the domain where $W \neq 0$, the $*$ represents complex conjugate, the index n represents an enumeration of the basis functions, and δ_m^n is the Kronecker delta. To proceed one uses completeness to write

$$f(\tau, v) = \sum_{m=1}^{\infty} a_m g_m(\tau, v), \quad (3)$$

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where the a_m s are unknown coefficients.

One proceeds by multiplying equation (1) by $g_n^*(\tau, v)W(\tau, v)d\tau dv$ and integrates to obtain

$$\lambda a_n = \sum_{m=1}^{\infty} T_{n,m} a_m, \quad (4)$$

where the matrix elements $T_{n,m}$ depend on the basis chosen, the impedance, etc. If L_{op} is sufficiently well behaved the technique seems to be exact to this point. In practical applications the infinite matrix equation is intractable, the sum is truncated at some value N , and eigenmodes are obtained using numerical techniques[3]. There is circumstantial evidence[9] that truncating the sum can lead to grossly incorrect eigenvalues even for large N .

The purpose of this paper is to examine the expansion technique using a very simple, exactly solvable model of bunched beam stability. The model is quite naive, but seems to incorporate the fundamental elements of a longitudinal instability calculation. It is found that one must be careful in drawing conclusions from the basis expansion results.

2 Longitudinal instability model

The model assumes a waterbag distribution in a square well longitudinal potential with an impedance $Z = R - i\omega L$. The particles undergo perfect reflection at the edges of the bunch which makes the rf restoring force a boundary condition[8, 9, 10].

Let θ denote machine azimuth, ω_0 be the angular revolution frequency of a synchronous particle, and τ be arrival time relative to the head of the bunch. Using θ as the time-like variable the Vlasov equation away from the reflective boundaries is,

$$\frac{\partial f}{\partial \theta} + v \frac{\partial f}{\partial \tau} - \kappa \left(IR + L \frac{\partial I}{\partial \tau} \right) \frac{\partial f}{\partial v} = 0. \quad (5)$$

In equation (5): $f = f(\theta, \tau, v)$, $I = I(\tau, \theta) = q \int dv f(\theta, \tau, v)$, $v = d\tau/d\theta$, and

$$\kappa = \frac{\eta}{2\pi\omega_0\beta^2(E_0/q)},$$

with slip factor η . The synchronous particle's energy, charge, and velocity are E_0 , q , and βc , respectively. For a waterbag distribution below the wave-breaking threshold the solution to the Vlasov equation is of the form:

$$f(\theta, \tau, v) = f_0 H(v_+(\theta, \tau) - v) H(v - v_-(\theta, \tau)) H(\tau) H(\tau_b - \tau) \quad (6)$$

where f_0 is a constant and $H(x)$ is the heaviside function with

$$H(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

Substituting equation (6) into (5) results in differential equations for v_+ and v_- ,

$$\frac{\partial v_{\pm}}{\partial \theta} + v_{\pm} \frac{\partial v_{\pm}}{\partial \tau} = F,$$

with

$$F = -\kappa \left(IR + L \frac{\partial I}{\partial \tau} \right).$$

The current is proportional to $v_+ - v_-$ and the solution is exact to this point. Next, neglect the effect of R in the unperturbed distribution setting

$$v_{\pm} = \pm \hat{v} + \delta v_{\pm}(\tau) \exp(-iQ\theta).$$

Setting $\delta v_+(\tau) + \delta v_-(\tau) = D(\tau)$ yields a single equation for D with

$$Q^2 D = -\hat{v}(\hat{v} - 2V) \frac{d^2 D}{d\tau^2} + 2U\hat{v} \frac{dD}{d\tau}, \quad (8)$$

where $U - iV = -q\kappa\omega_0 f_0(R - i\omega_0 L)$. For perfect reflection at $\tau = 0$ and $\tau = \tau_b$ the boundary conditions are $D(0) = D(\tau_b) = 0$. To solve (8) notice

that

$$D(\tau) = \exp(\lambda_+ \tau) - \exp(\lambda_- \tau).$$

Inserting this expression yields a quadratic equation for λ_{\pm} ,

$$\lambda_{\pm} = \frac{U}{\hat{v} - 2V} \pm \sqrt{\frac{U^2}{(\hat{v} - 2V)^2} - \frac{Q^2}{\hat{v}(\hat{v} - 2V)}}.$$

The boundary condition at τ_b gives $\lambda_+ - \lambda_- = 2\pi i k / \tau_b$ with $k \neq 0$ an integer. The eigenvalue satisfies

$$Q_k^2 = \hat{v}(\hat{v} - 2V) \left\{ \frac{k^2 \pi^2}{\tau_b^2} + \frac{U^2}{(\hat{v} - 2V)^2} \right\}, \quad (9)$$

and the eigenvector is

$$D_k(\tau) = \sin(k\pi\tau/\tau_b) \exp\left(\tau \frac{U}{\hat{v} - 2V}\right). \quad (10)$$

The functions D_k for $k = 1, 2, \dots$ form a complete set on $(0, \pi)$, so no eigenmodes have been missed. The right hand side of (9) is positive as long as $\hat{v} > 2V$ or

$$-\frac{qI_0\eta}{2\pi E_0\beta^2}\omega_0 L < \hat{v}^2.$$

Resistance alone cannot cause instability.

While the reader may disagree with the physical behavior, the fact that equations (9) and (10) represent the complete solution of (8) is inescapable. In fact, everything before equation (8) may be viewed as purely motivational without affecting the main results.

3 Testing basis expansions

Consider equation (8) with $\hat{v}(\hat{v}-2V) > 0$. Let $x = \pi\tau/\tau_b$, $\nu^2 = Q^2\tau_b^2/\pi^2\hat{v}(\hat{v}-2V)$, and $\tilde{R} = U\tau_b/\pi(\hat{v} - 2V)$. The equation becomes

$$\nu^2 D = -\frac{d^2 D}{dx^2} + 2\tilde{R}\frac{dD}{dx}, \quad (11)$$

with boundary conditions $D(0) = D(\pi) = 0$. In these variables the exact eigenvectors and eigenvalues are

$$D_k(x) = \sin(kx) \exp(\tilde{R}x), \quad (12)$$

$$\nu_k^2 = k^2 + \tilde{R}^2. \quad (13)$$

Let the expansion functions be $g_n(x) = \sin(nx)$ with $W = 2/\pi$ for $0 < x < \pi$. The eigenvalue in equation (4) is $\lambda = Q^2$ and the matrix element is

$$T_{n,m} = nm\delta_m^n + \frac{8\tilde{R}nm}{\pi(n^2 - m^2)}\text{odd}(n - m), \quad (14)$$

where $\text{odd}(n - m) = 1$ if $n - m$ is odd, and $\text{odd}(n - m) = 0$ otherwise. A computer code was used to find solutions to

$$Q^2 a_n = \sum_{m=1}^N T_{n,m} a_m, \quad (15)$$

for various values of N and \tilde{R} . The only sophisticated part of the code is the eigenvalue solver[11] which is a well tested standard routine. Figure 1 shows the values of Q as a function \tilde{R} for $N = 2$ and predicts instability for $\tilde{R} \gtrsim 1$. Setting $N = 9$ gives Figure 2. The lowest frequency mode in Figure 2 is stable, this appears to be the case for all odd N . Also notice that large values of Q go unstable first. This also seems to be generic. Setting $\tilde{R} \rightarrow -\tilde{R}$ has no effect on the eigenvalue spectrum, so crossing transition is not an issue.

As the number of modes increases the threshold value of \tilde{R} for coupling between the lowest order modes continues to increase. Figure 3 shows the lowest 10 modes for $N = 100$. The expansion technique predicts that lowest order modes go unstable for $\tilde{R} < 5$, while the exact solution proves that the system is absolutely stable. Additionally, the threshold value of \tilde{R} for the onset of unstable behavior without regard to mode varies only slightly with N . The threshold values of \tilde{R} are 0.90, 0.95, and 0.95 for $N = 2$, 9, and 100, respectively. With basis expansions alone could one show that equation (11) has no unstable solutions?

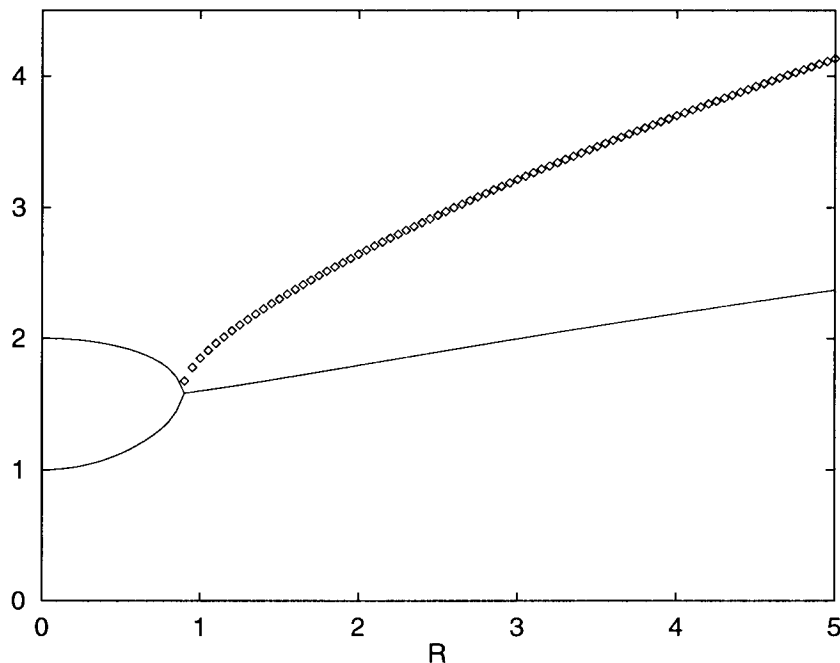


Figure 1: Real and imaginary parts of Q versus \tilde{R} for $N = 2$. The solid lines are the real part of Q and the distance between the points and solid line is the imaginary part of Q for the unstable mode.

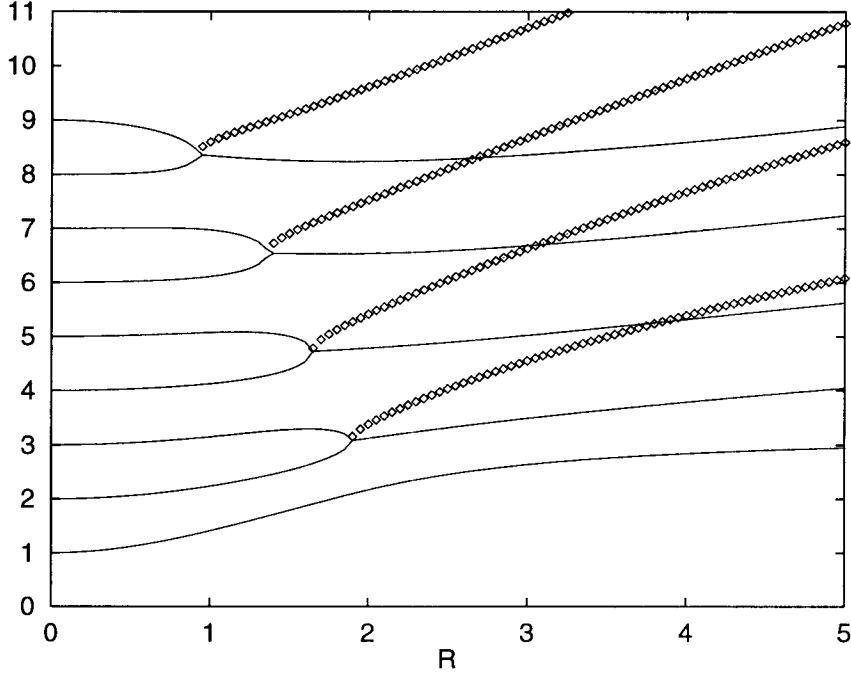


Figure 2: Real and imaginary parts of Q versus \tilde{R} for $N = 9$. The solid lines are the real part of Q and the distance between the points and solid line is the imaginary part of Q for the unstable mode.

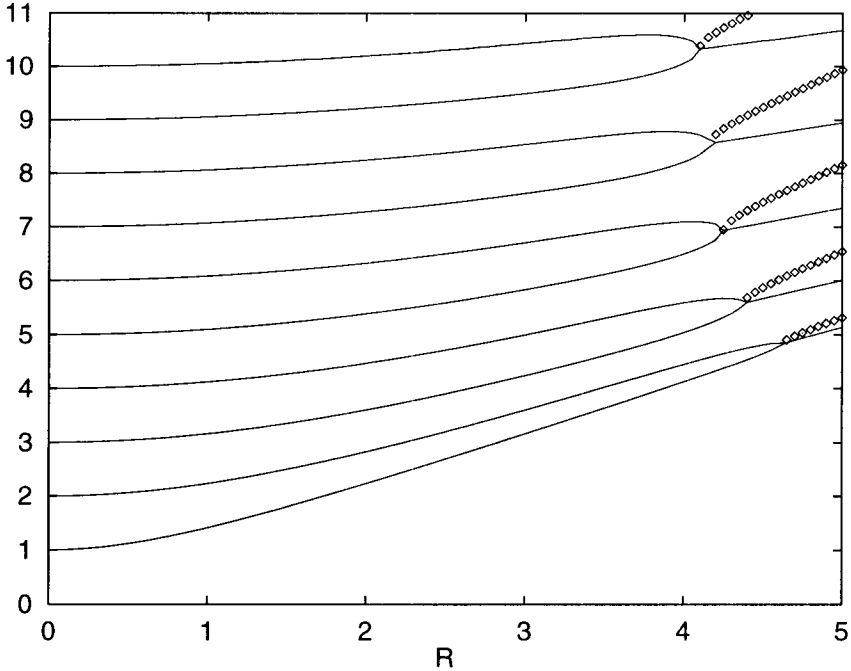


Figure 3: Lowest 10 eigenvalues versus \tilde{R} for $N = 100$. The solid lines are the real part of Q and the distance between the points and solid line is the imaginary part of Q for the unstable mode.

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